Markscheme

May 2016

Further mathematics

Higher level

Paper 2

18 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2016**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).
A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) the graph is not Eulerian *A1*
because the graph contains vertices of odd degree *R1*
[2 marks]

(b) the graph is Hamiltonian *A1*
because, for example, ABDFHGECA is a Hamiltonian cycle *R1*
[2 marks]

(c) correctly start to use Kruskal's algorithm DE(1) ***(M1)***
BC(2), FG(2) or vice-versa *A1*
DC(3), AC(3) or vice-versa *A1*
GH(4) (rejecting AB) *A1*
DF(5) or EG(5) (rejecting BD) *A1*
total weight $= 20$ *A1*
[6 marks]

(d) the minimum weight spanning tree can be traversed twice ***(M1)***
so upper bound is $2 \times 20 = 40$ *A1*
[2 marks]

(e) the Hamiltonian cycle found in (b) is a closed walk visiting every vertex and hence can be applied here *R1*
weight $= 39$ *A1*
[2 marks]

Total [14 marks]

2. (a) attempt to apply l'Hôpital's rule *M1*

$\lim_{x \to \infty} \frac{3x^2}{e^x}$ *A1*

then $\lim_{x \to \infty} \frac{6x}{e^x}$

then $\lim_{x \to \infty} \frac{6}{e^x}$ *A1*

$= 0$ *AG*

[3 marks]

continued…

Question 2 continued

(b) (i) $\mathrm{E}(X^2) = \lim_{R\to\infty}\int_0^R x^3 \mathrm{e}^{-x}\mathrm{d}x$ ***M1***

attempt at integration by parts ***M1***

the integral $= \left[-x^3\mathrm{e}^{-x}\right]_0^R + \int_0^R 3x^2\mathrm{e}^{-x}\mathrm{d}x$ ***A1A1***

$= \left[-x^3\mathrm{e}^{-x}\right]_0^R + \left[-3x^2\mathrm{e}^{-x}\right]_0^R + \int_0^R 6x\mathrm{e}^{-x}\,\mathrm{d}x$ ***M1***

$= \left[-x^3\mathrm{e}^{-x}\right]_0^R + \left[-3x^2\mathrm{e}^{-x}\right]_0^R + \left[-6x\mathrm{e}^{-x}\right]_0^R + \int_0^R 6\mathrm{e}^{-x}\,\mathrm{d}x$ ***A1***

$= \left[-x^3\mathrm{e}^{-x}\right]_0^R + \left[-3x^2\mathrm{e}^{-x}\right]_0^R + \left[-6x\mathrm{e}^{-x}\right]_0^R + \left[-6\mathrm{e}^{-x}\right]_0^R$ ***A1***

$= 6$ when $R \to \infty$ ***R1***

(ii) $\mathrm{E}(X) = 2$ ***A1***

$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 6 - 2^2$ ***M1***

$= 2$ ***AG***

[10 marks]

(c) if a random sample of size n is taken from ***any*** distribution X, with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, then, for ***large n***, ***A1***

the sample mean $\bar{X}$ has approximate distribution $\mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ ***A1***

[2 marks]

(d) $\bar{X} \sim \mathrm{N}\left(2, \frac{2}{50} = (0.2)^2\right)$ ***(A1)***

$\mathrm{P}(\bar{X} < 2.3) = (\mathrm{P}(Z < 1.5)) = 0.933$ ***A1***

[2 marks]

Total [17 marks]

3. (a) **METHOD 1**

attempt to exploit the fact that the normal to a tangent passes through the centre (a, b) ***(M1)***

EITHER

equation of normal is $y - 4 = -\frac{1}{3}(x - 3)$ ***(A1)***

obtain $a + 3b = 15$ ***A1***

attempt to exploit the fact that a circle has a constant radius: ***(M1)***

obtain $(1 - a)^2 + (2 - b)^2 = (3 - a)^2 + (4 - b)^2$ ***A1***

leading to $a + b = 5$ ***A1***

centre is $(0, 5)$ ***(M1)A1***

radius $= \sqrt{1^2 + 3^2} = \sqrt{10}$ ***A1***

OR

gradient of normal $= -\frac{1}{3}$ ***A1***

general point on normal $= (3 - 3\lambda, 4 + \lambda)$ ***(M1)A1***

this point is equidistant from $(1, 2)$ and $(3, 4)$ ***M1***

if $10\lambda^2 = (2 - 3\lambda)^2 + (2 + \lambda)^2$

$10\lambda^2 = 4 - 12\lambda + 9\lambda^2 + 4 + 4\lambda + \lambda^2$ ***A1***

$\lambda = 1$ ***A1***

centre is $(0, 5)$ ***A1***

radius $= \sqrt{10\lambda} = \sqrt{10}$ ***A1***

METHOD 2

attempt to substitute two points in the equation of a circle ***(M1)***

$(1 - h)^2 + (2 - k)^2 = r^2$, $(3 - h)^2 + (4 - k)^2 = r^2$ ***A1***

Note: The ***A1*** is for the two LHSs, which may be seen equated.

equate or subtract the equations

obtain $h + k = 5$ or equivalent ***A1***

attempt to differentiate the circle equation implicitly ***(M1)***

obtain $2(x - h) + 2(y - k)\frac{dy}{dx} = 0$ ***A1***

Note: Similarly, ***M1A1*** if direct differentiation is used.

substitute $(3, 4)$ and gradient $= 3$ to obtain $h + 3k = 15$ ***A1***

obtain centre $= (0, 5)$ ***(M1)A1***

radius $= \sqrt{10}$ ***A1***

[9 marks]

continued…

Question 3 continued

(b) equation of circle is $x^2 + (y - 5)^2 = 10$ ***A1***

[1 mark]

(c) the equation of the chord is $3x - y = 1$ ***A1***

attempt to solve the equation for the chord and that for the circle simultaneously ***(M1)***

for example $x^2 + (3x - 1 - 5)^2 = 10$ ***A1***

coordinates of the second point are $\left(\frac{13}{5}, \frac{34}{5}\right)$ ***(M1)A1***

[5 marks]

Total [15 marks]

4. (a) attempt to separate the variables ***M1***

$\int y \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int x \, \mathrm{d}x$ ***A1***

Note: Accept $\int y \, \mathrm{d}y = \int x \, \mathrm{d}x$.

obtain $\frac{1}{2} y^2 = \frac{1}{2} x^2 +$ constant $\left(\Rightarrow y^2 - x^2 = c \right)$ ***A1***

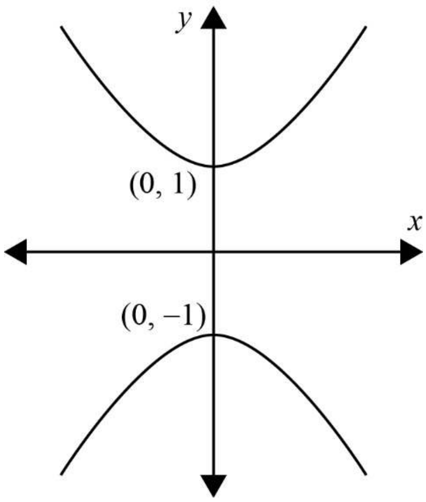
[3 marks]

(b) (i) substitute the coordinates for both points ***M1***

$\left(\pm \sqrt{2} \right)^2 - 1^2 = 1$

obtain $y^2 - x^2 = 1$ or equivalent ***A1***

(ii)



A1A1

Note: ***A1*** for general shape including two branches and symmetry;
A1 for values of the intercepts.

(rectangular) hyperbola ***A1***

[5 marks]

(c) (i) $y^2 - x^2 = 0$ ***A1***

(ii) the two straight lines $y = \pm x$ ***A1***

form the asymptotes to the hyperbola found above, or equivalent ***A1***

[3 marks]

continued…

Question 4 continued

(d) (i) the equation is homogeneous, so attempt to substitute $y = vx$ ***M1***

as a first step write $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}v}{\mathrm{d}x} + v$ ***(A1)***

then $x\frac{\mathrm{d}v}{\mathrm{d}x} + v = \frac{1}{v} + v$ ***A1***

attempt to solve the resulting separable equation ***M1***

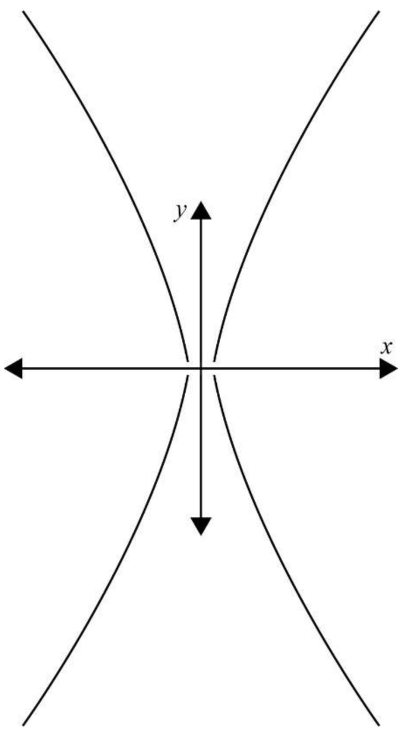
$\int v\,\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$ ***A1***

obtain $\frac{1}{2}v^2 = \ln|x| +$ constant $\Rightarrow y^2 = 2x^2\ln|x| + cx^2$ ***A1***

(ii) substituting the coordinates ***(M1)***

obtain $c = 2 \Rightarrow y^2 = 2x^2\ln|x| + 2x^2$ ***A1***

(iii)



A1

(iv) since $y^2 > 0$ and $x^2 \neq 0$ ***R1***

$\ln|x| > -1 \Rightarrow |x| > \mathrm{e}^{-1}$ ***A1***

$a = \mathrm{e}^{-1}$ ***A1***

Note:	The ***R1*** may be awarded for a correct reason leading to subsequent correct work.

[12 marks]

Total [23 marks]

5. (a) (i) the auxiliary equation is $2r^2 - 3r + 1 = 0$ *(M1)*

with roots $r = 1, \frac{1}{2}$ *A1*

the general solution of the difference equation is *(M1)*

$u_n = A + B\left(\frac{1}{2}\right)^n$ *A1*

imposing the initial conditions *M1*

$A + \frac{B}{2} = 1,\ A + \frac{B}{4} = 2$ *A1*

obtain $u_n = 3 - 4\left(\frac{1}{2}\right)^n$ *A1*

(ii) as $n \to \infty, \left(\frac{1}{2}\right)^n \to 0$ *R1*

$u_n \to 3$ *A1*

hence the sequence is convergent *AG*

[9 marks]

(b) assume $v_n \to L$ *M1*

taking the limit of both sides of the recurrence relation *M1*

$2L - 3L + L\ (= 0) = 1$ *A1*

the contradiction shows that the sequence diverges *AG*

[3 marks]

(c) (i) the auxiliary equation $r^2 - 2r + 4 = 0$ *A1*

has roots $1 \pm \mathrm{i}\sqrt{3}$ *A1*

METHOD 1

these can be re-expressed as $2\left(\cos\left(\frac{\pi}{3}\right) \pm \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$ *M1*

the general solution is

$w_n = 2^n\left(A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right)\right)$ *A1*

imposing the initial conditions

$A = 0,\ 2B\frac{\sqrt{3}}{2} = 2$ *A1*

obtain $w_n = \frac{2^{n+1}}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right)$ *A1*

continued…

Question 5 continued

METHOD 2

the general solution is

$w_n = A\left(1+\mathrm{i}\sqrt{3}\right)^n + B\left(1-\mathrm{i}\sqrt{3}\right)^n$ ***A1***

imposing the initial conditions

$A + B = 0,\ A + B + \mathrm{i}\sqrt{3}(A - B) = 2$ ***M1A1***

obtain $w_n = \frac{1}{\mathrm{i}\sqrt{3}}\left(1+\mathrm{i}\sqrt{3}\right)^n - \frac{1}{\mathrm{i}\sqrt{3}}\left(1-\mathrm{i}\sqrt{3}\right)^n$ ***A1***

(ii) **METHOD 1**

$w_{3n} = \frac{2^{3n+1}}{\sqrt{3}}\sin(n\pi)$ ***R1***

$=0$ ***AG***

METHOD 2

$w_{3n} = \frac{1}{\mathrm{i}\sqrt{3}}\left(1+\mathrm{i}\sqrt{3}\right)^{3n} - \frac{1}{\mathrm{i}\sqrt{3}}\left(1-\mathrm{i}\sqrt{3}\right)^{3n}$

$= \frac{1}{\mathrm{i}\sqrt{3}}(-8)^n - \frac{1}{\mathrm{i}\sqrt{3}}(-8)^n$ ***R1***

$=0$ ***AG***

[7 marks]

Total [19 marks]

6. (a) $\left(a+b\sqrt{2}\right)\times\left(c+d\sqrt{2}\right) = ac + bc\sqrt{2} + ad\sqrt{2} + 2bd$ ***M1***

$= ac + 2bd + (bc + ad)\sqrt{2} \in J$ ***A1***

hence J is closed ***AG***

Note: Award ***M0A0*** if the general element is squared.

[2 marks]

(b) the identity is $1(a = 1,\ b = 0)$ ***A1***

[1 mark]

continued…

Question 6 continued

(c) (i) $\left(1-\sqrt{2}\right)\times a=1$

$a=\dfrac{1}{1-\sqrt{2}}$ ***M1***

$=\dfrac{1+\sqrt{2}}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}=\dfrac{1+\sqrt{2}}{-1}=-1-\sqrt{2}$ ***A1***

hence $1-\sqrt{2}$ has an inverse in J ***AG***

(ii) $\left(2+4\sqrt{2}\right)\times a=1$

$a=\dfrac{1}{2+4\sqrt{2}}$ ***M1***

$=\dfrac{2-4\sqrt{2}}{\left(2-4\sqrt{2}\right)\left(2+4\sqrt{2}\right)}=\dfrac{2-4\sqrt{2}}{-28}$ ***A1***

which does not belong to J ***R1***

hence $2+4\sqrt{2}$ has no inverse in J ***AG***

[5 marks]

(d) multiplication is associative ***A1***

let g_1 and g_2 belong to G, then g_1^{-1}, g_2^{-1} and $g_2^{-1}g_1^{-1}$ belong to J ***M1***

then $\left(g_1 g_2\right)\times\left(g_2^{-1}g_1^{-1}\right)=1\times 1=1$ ***A1***

so $g_1 g_2$ has inverse $g_2^{-1}g_1^{-1}$ in $J\Rightarrow G$ is closed ***A1***

G contains the identity ***A1***

G possesses inverses ***A1***

G contains all integral powers of $1-\sqrt{2}$ ***A1***

hence G is an infinite group ***AG***

[7 marks]

(e) (i) $\left(a+b\sqrt{2}\right)^{-1}=\dfrac{1}{a+b\sqrt{2}}=\dfrac{1}{a+b\sqrt{2}}\times\dfrac{a-b\sqrt{2}}{a-b\sqrt{2}}$ ***M1***

$=\dfrac{a}{a^2-2b^2}-\dfrac{b}{a^2-2b^2}\sqrt{2}$ ***A1***

(ii) above number belongs to J and $a^2-2b^2\in\mathbb{Z}$ ***R1***

implies a^2-2b^2 divides exactly into a and b ***AG***

(iii) since $\gcd(a,b)=1$ ***R1***

$a^2-2b^2=\pm 1$ ***AG***

[4 marks]

Total [19 marks]

7. (a) (i) $\frac{\mathrm{d}f_n(x)}{\mathrm{d}x} = n\sec^{n-1}(x)\sec(x)\tan(x)$ ***M1A1***

$= ng_n(x)$ ***AG***

(ii) $\frac{\mathrm{d}g_n(x)}{\mathrm{d}x} = \frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\tan(x) + f_n(x)\sec^2(x)$ ***M1***

$ng_n(x)\tan(x) + f_{n+2}(x)$ or equivalent ***A1***

$nf_n(x)\tan^2(x) + f_{n+2}(x)$ or equivalent ***A1***

$= (n+1)f_{n+2}(x) - nf_n(x)$ ***AG***

Note: Award ***M1A1*** for the correct differentiation of a product and ***A1*** for an intermediate result clearly leading to the ***AG***.

[5 marks]

(b) (i) $f_5(0) = 1$ ***A1***

$\frac{\mathrm{d}f_5}{\mathrm{d}x}(0) = 5g_5(0) = 0$ ***A1***

$\frac{\mathrm{d}^2 f_5}{\mathrm{d}x^2}(0) = 5\left(6f_7(0) - 5f_5(0)\right) = 5$ ***A1***

$\frac{\mathrm{d}^3 f_5}{\mathrm{d}x^3} = 30\frac{\mathrm{d}f_7}{\mathrm{d}x} - 25\frac{\mathrm{d}f_5}{\mathrm{d}x}$ ***M1***

hence $\frac{\mathrm{d}^3 f_5}{\mathrm{d}x^3}(0) = 30 \times 0 - 25 \times 0 = 0$ ***A1***

$\frac{\mathrm{d}^4 f_5}{\mathrm{d}x^4} = 30\frac{\mathrm{d}^2 f_7}{\mathrm{d}x^2} - 25\frac{\mathrm{d}^2 f_5}{\mathrm{d}x^2} = 210(8f_9 - 7f_7) - 25\frac{\mathrm{d}^2 f_5}{\mathrm{d}x^2}$ ***M1A1***

hence $\frac{\mathrm{d}^4 f_5}{\mathrm{d}x^4}(0) = 210 - 125 = 85$ ***A1***

hence $f_5(x) \approx 1 + \frac{5}{2}x^2 + \frac{85}{24}x^4$ ***AG***

(ii) each derivative of $f_m(x)$ is a sum of terms of the form $\mathrm{A}\sec^p(x)\tan^q(x)$ ***A1***

where $\mathrm{A} \geq 0$ ***A1***

when $x = 0$ is substituted the result is the sum of positive and/or zero terms ***R1***

(iii) since the full series represents $f_5(x)$, the truncated series is a lower bound (or some equivalent statement) ***R1***

hence $\sec^5(0.1) > 1 + \frac{5}{2}0.1^2 + \frac{85}{24}0.1^4$ ***M1***

$= 1.025354$ ***A1***

> 1.02535 ***AG***

[14 marks]

Total [19 marks]

8. (a) (i) 1 has n possible new positions; 2 then has $n-1$ possible new positions…
n has only one possible new position ***R1***
the number of possible permutations is $n \times (n-1) \times \ldots \times 2 \times 1$ ***R1***
$= n!$ ***AG***

Note: Give no credit for simply stating that the number of permutations is $n!$

(ii) $(1)(2)(3)$; $(1\ 2)(3)$; $(1\ 3)(2)$; $(2\ 3)(1)$; $(1\ 2\ 3)$; $(1\ 3\ 2)$ ***A2***

Notes: ***A1*** for 4 or 5 correct. If single bracket terms are missing, do not penalize. Accept e in place of the identity.

(iii) attempt to compare $\pi_1 \circ \pi_2$ with $\pi_2 \circ \pi_1$ for two permutations ***M1***
for example $(1\ 2)(1\ 3) = (1\ 3\ 2)$ ***A1***
but $(1\ 3)(1\ 2) = (1\ 2\ 3)$ ***A1***
hence S_3 is not Abelian ***AG***

(iv) S_3 is a subgroup of S_n, ***R1***
so S_n contains non-commuting elements ***R1***
$\Rightarrow S_n$ is not Abelian for $n \geq 3$ ***AG***

[9 marks]

(b) (i) $\boldsymbol{M}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ***A1A1***

(ii) $\boldsymbol{M}_1\boldsymbol{M}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ***A1***

this represents $(1\ 3\ 2)$ ***A1***

(iii) by, for example, interchanging a pair of rows ***(M1)***
$\det(\boldsymbol{M}_1) = \det(\boldsymbol{M}_2) = -1$ ***A1***
then $\det(\boldsymbol{M}_1\boldsymbol{M}_2) = (-1) \times (-1) = 1$ ***A1***

[7 marks]

continued…

Question 8 continued

(c) (i) let $\mathrm{P}(n)$ be the proposition that

$(1\ \ n)(1\ \ n-1)(1\ \ n-2)\ldots(1\ 2) = (1\ 2\ 3\ldots n)\ n\in\mathbb{Z}^+$

the statement that $\mathrm{P}(2)$ is true *eg* $(1\ 2) = (1\ 2)$ ***A1***

assume $\mathrm{P}(k)$ is true for some k ***M1***

consider $(1\ \ k+1)(1\ \ k)(1\ \ k-1)(1\ \ k-2)\ldots(1\ 2)$

$= (1\ \ k+1)(1\ 2\ 3\ldots k)$ ***M1***

then the composite permutation has the following effect on the first $k+1$ integers: $1\to 2,\ 2\to 3\ldots\ k-1\to k,\ k\to 1\to k+1,\ k+1\to 1$ ***A1***

this is $(1\ 2\ 3\ \ldots k\ k+1)$ ***A1***

hence the assertion is true by induction ***AG***

(ii) every permutation is a product of cycles ***R1***

generalizing the result in (i) ***R1***

every cycle is a product of cycles of length 2 ***R1***

hence every permutation can be written as a product of cycles of length 2 ***AG***

[8 marks]

Total [24 marks]